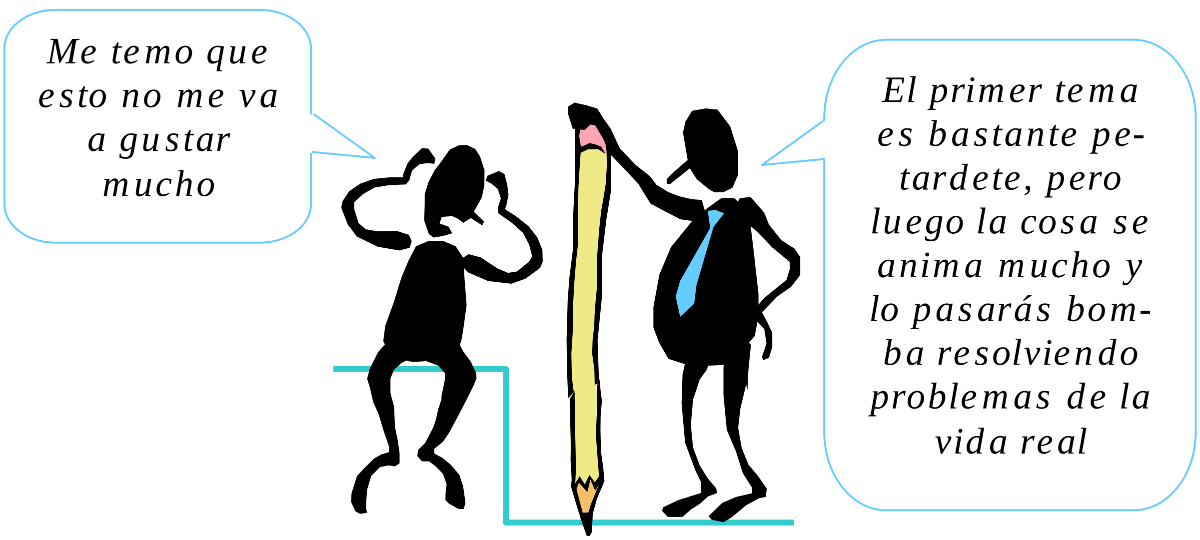
Tema 1

Cálculo de primitivas



1.6 CÁLCULO DE PRIMITIVAS "POR PARTES"

Si "u" y "v" son funciones de la variable "x", es $d(u.v) = v.du + u.dv$; al despejar $u.dv$ resulta $u.dv = d(u.v) - v.du$, y así:

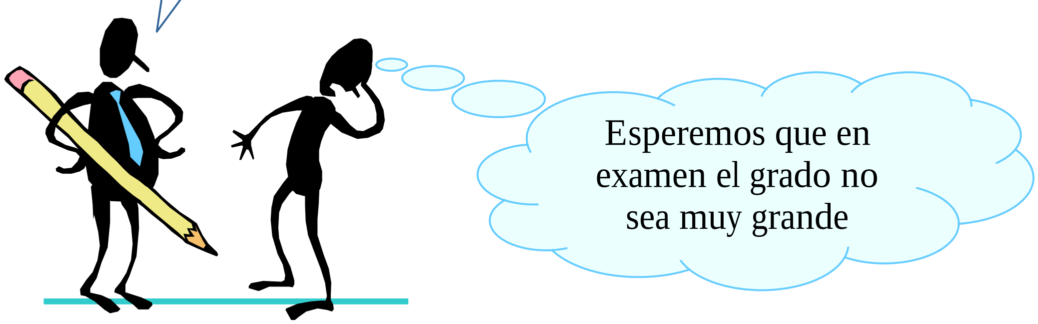
$$\int u.dv = \int d(u.v) - \int v.du$$

Como $\int d(u.v) = u.v$, resulta la llamada fórmula de integración por partes

$$\int u.dv = u.v - \int v.du$$

que transforma el problema de calcular $\int u.dv$ en el problema de calcular $\int v.du$

En los siguientes seis ejemplos calculamos primitivas de funciones cuya "estructura" es $f(x) = P(x).a^{b.x}$ $(a > 0)$, siendo $P(x)$ un polinomio y "b" una constante. Además, para aplicar la formulita $\int u.dv = u.v - \int v.du)$ siempre haremos $u = P(x)$ y $dv = a^{b.x}.dx$, repitiendo el proceso tantas veces como grado tiene el polinomio.



$\boxed{1.6.1}$

$$\int x.e^x.dx = x.e^x - \int e^x.dx = x.e^x - e^x + C$$

$$u = x \Rightarrow du = dx \quad \Big| \quad dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$\boxed{1.6.2}$

$$\int x^2.e^x.dx = x^2.e^x - 2.\int x.e^x.dx = x^2.e^x - 2.(x.e^x - \int e^x.dx) =$$

$$* \; u = x^2 \Rightarrow du = 2.x.dx$$
$$* \; dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$$* \; u = x \Rightarrow du = dx$$
$$* \; dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$$= x^2.e^x - 2.x.e^x + 2.e^x + C$$

$\boxed{1.6.3}$

$$\int x^3.e^x.dx = x^3.e^x - 3.\int x^2.e^x.dx =$$

$$* \; u = x^3 \Rightarrow du = 3.x^2.dx$$
$$* \; dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$$= x^3.e^x - 3.(x^2.e^x - 2.\int x.e^x.dx) =$$

$$* \; u = x^2 \Rightarrow du = 2.x.dx$$
$$* \; dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$$= x^3.e^x - 3.x^2.e^x + 6.(x.e^x - \int e^x.dx) =$$

$$* \ u = x \Rightarrow du = dx$$
$$* \ dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$$= x^3.e^x - 3.x^2.e^x + 6.x.e^x - 6.e^x + C$$

1.6.4 $$\int x^2.3^x.dx = \frac{x^2.3^x}{Ln\ 3} - \frac{2}{Ln\ 3}.\int x.3^x.dx =$$

$$* \ u = x^2 \Rightarrow du = 2.x.dx$$
$$* \ dv = 3^x.dx \Rightarrow v = \int 3^x.dx = \frac{3^x}{Ln\ 3}$$

$$= \frac{x^2.3^x}{Ln\ 3} - \frac{2}{Ln\ 3}.(\frac{x.3^x}{Ln\ 3} - \frac{1}{Ln\ 3}.\int 3^x.dx) =$$

$$* \ u = x \Rightarrow du = dx$$
$$* \ dv = 3^x.dx \Rightarrow v = \int 3^x.dx = \frac{3^x}{Ln\ 3}$$

$$= \frac{x^2.3^x}{Ln\ 3} - \frac{2.x.3^x}{(Ln\ 3)^2} - \frac{3^x}{(Ln\ 3)^3} + C$$

1.6.5 $$\int x^2.e^{3.x}.dx = \frac{x^2.e^{3.x}}{3} - \frac{2}{3}.\int x.e^{3.x}.dx =$$

$$* \ u = x^2 \Rightarrow du = 2.x.dx$$
$$* \ dv = e^{3.x}.dx \Rightarrow v = \int e^{3.x}.dx = e^x/3$$

$$= \frac{x^2.e^{3.x}}{3} - \frac{2}{3}.(\frac{x.e^{3.x}}{3} - \frac{1}{3}.\int e^{3.x}.dx) =$$

$$* \ u = x \Rightarrow du = dx$$
$$* \ dv = e^{3.x}.dx \Rightarrow v = \int e^{3.x}.dx = e^x/3$$

$$= \frac{x^2.e^{3.x}}{3} - \frac{2.x.e^{3.x}}{9} + \frac{2.e^{3.x}}{27} + C$$

1.6.6 $$\int x^2.e^{-x}.dx = -x^2.e^{-x} + 2.\int x.e^{-x}.dx =$$

$$* \ u = x^2 \Rightarrow du = 2.x.dx$$
$$* \ dv = e^{-x}.dx \Rightarrow v = \int e^{-x}.dx = -e^x$$

$$= -x^2.e^{-x} + 2.(-x.e^{-x} + \int e^{-x}.dx) =$$

$$* \ u = x \Rightarrow du = dx$$
$$* \ dv = e^{-x}.dx \Rightarrow v = \int e^{-x}.dx = -e^x$$

$$= -x^2.e^{-x} - 2.x.e^{-x} - 2.e^{-x} + C$$

En los siguientes seis ejemplos calculamos primitivas de funciones cuya "estructura" es $f(x) = P(x).\text{sen ax}$ ó $f(x) = P(x).\cos ax$, siendo $P(x)$ un polinomio y "a" una constante. Para aplicar la formulita $\int u.dv = u.v - \int v.du$ haremos $u = P(x)$ y $dv = \text{sen ax}$ ó $u = P(x)$ y $dv = \cos ax$, repitiendo el proceso tantas veces como grado tiene el polinomio.

$\boxed{1.6.7}$ $\int x.\text{sen } x.dx = -x.\cos x + \int \cos x.dx = -x.\cos x + \text{sen } x + C$

$$\begin{array}{c} * \; u = x \Rightarrow du = dx \\ * \; dv = \text{sen } x.dx \Rightarrow v = \int \text{sen } x.dx = -\cos x \end{array}$$

$\boxed{1.6.8}$ $\int x.\cos x.dx = x.\text{sen } x - \int \text{sen } x.dx = x.\cos x + \cos x + C$

$$\begin{array}{c} * \; u = x \Rightarrow du = dx \\ * \; dv = \cos x.dx \Rightarrow v = \int \cos x.dx = \text{sen } x \end{array}$$

$\boxed{1.6.9}$ $\int x.\text{sen } 4.x.dx = -\dfrac{x.\cos 4.x}{4} + \dfrac{1}{4}.\int \cos 4.x.dx =$

$$\begin{array}{c} * \; u = x \Rightarrow du = dx \\ * \; dv = \text{sen } 4.x.dx \Rightarrow v = \int \text{sen } 4.x.dx = -\dfrac{\cos 4.x}{4} \end{array}$$

$$= -\frac{x.\cos 4.x}{4} + \frac{\text{sen } 4.x}{16} + C$$

$\boxed{1.6.10}$ $\int x.\cos 5.x.dx = \dfrac{x.\text{sen } 5.x}{5} - \dfrac{1}{5}.\int \text{sen } 5.x.dx =$

$$\begin{array}{c} * \; u = x \Rightarrow du = dx \\ * \; dv = \cos 5.x.dx \Rightarrow v = \int \cos 5.x.dx = \dfrac{\text{sen } 5.x}{5} \end{array}$$

$$= \frac{x.\text{sen } 5.x}{5} + \frac{\cos 5.x}{25} + C$$

$\boxed{1.6.11}$ $\int x^2.\cos 3.x.dx = \dfrac{x^2.\text{sen } 3.x}{3} - \dfrac{2}{3}.\int x.\text{sen } 3.x.dx =$

$$\begin{array}{c} * \; u = x^2 \Rightarrow du = 2.x.dx \\ * \; dv = \cos 3.x.dx \Rightarrow v = \int \cos 3.x.dx = \dfrac{\text{sen } 3.x}{3} \end{array}$$

$$= \frac{x^2.\text{sen } 3.x}{3} - \frac{2}{3}.(-\frac{x.\cos 3.x}{3} + \frac{1}{3}.\int \cos 3.x.dx) =$$

$$* \ u = x \Rightarrow du = dx$$
$$* \ dv = \text{sen } 3.x.dx \Rightarrow v = \int \text{sen } 3.x.dx = -\frac{\cos 3.x}{3}$$

$$= \frac{x^2.\text{sen } 3.x}{3} + \frac{2.x.\cos 3.x}{9} - \frac{2.\text{sen } 3.x}{27} + C$$

1.6.12 $$\int x^3.\cos 4.x.dx = \frac{x^3.\text{sen } 4.x}{4} - \frac{3}{4}.\int x^2.\text{sen } 4.x.dx =$$

$$* \ u = x^3 \Rightarrow du = 3.x^2.dx$$
$$* \ dv = \cos 4.x.dx \Rightarrow v = \int \cos 4.x.dx = \frac{\text{sen } 4.x}{4}$$

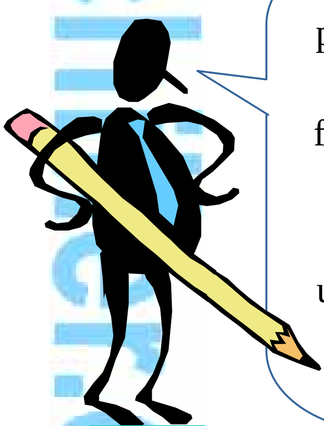
$$= \frac{x^3.\text{sen } 4.x}{4} - \frac{3}{4}.(-\frac{x^2.\cos 4.x}{4} + \frac{2}{4}.\int x.\cos 4.x.dx) =$$

$$* \ u = x^2 \Rightarrow du = 2.x.dx$$
$$* \ dv = \text{sen } 4.x.dx \Rightarrow v = \int \text{sen } 4.x.dx = -\frac{\cos 4.x}{4}$$

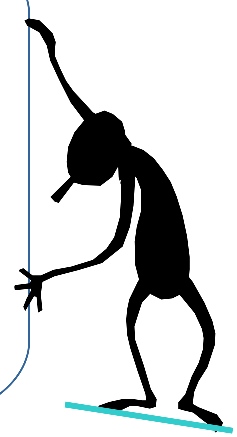
$$= \frac{x^3.\text{sen } 4.x}{4} + \frac{3.x^2.\cos 4.x}{4^2} - \frac{6}{4^2}.(\frac{x.\text{sen } 4.x}{4} - \frac{1}{4}.\int \text{sen } 4.x.dx) =$$

$$* \ u = x \Rightarrow du = dx$$
$$* \ dv = \cos 4.x.dx \Rightarrow v = \int \cos 4.x.dx = \frac{\text{sen } 4.x}{4}$$

$$= \frac{x^3.\text{sen } 4.x}{4} + \frac{3.x^2.\cos 4.x}{4^2} - \frac{6.x.\text{sen } 4.x}{4^3} + \frac{6.\cos 4.x}{4^4} + C$$



En los siguientes tres ejemplos calculamos primitivas de funciones cuya "estructura" es de la forma $f(x) = k^{a.x}.\text{sen } bx$ ó $f(x) = k^{a.x}.\cos bx$, donde $k > 0$ y "a" y "b" son constantes. Para aplicar la formulita $\int u.dv = u.v - \int v.du$ da lo mismo elegir $u = k^{a.x}$ ó $dv = k^{a.x}.dx$; tras repetir el proceso dos veces aparecerá de nuevo la primitiva que hemos de calcular.



1.6.13 $$\int e^x.\text{sen } x.dx = e^x.\text{sen } x - \int e^x.\cos x.dx =$$

$$* \ u = \text{sen } x \Rightarrow du = \cos x.dx$$
$$* \ dv = e^x.dx \Rightarrow v = \int e^x.dx = e^x$$

$$= e^{x}.\text{sen } x - (e^{x}.\cos x + \int e^{x}.\text{sen } x.dx)$$

$$\begin{array}{l} * \; u = \cos x \Rightarrow du = -\text{sen } x.dx \\ * \; dv = e^{x}.dx \Rightarrow v = \int e^{x}.dx = e^{x} \end{array}$$

O sea, aparece de nuevo la primitiva que debemos calcular:

$$\int e^{x}.\text{sen } x.dx = e^{x}.\text{sen } x - e^{x}.\cos x - \int e^{x}.\text{sen } x.dx$$

por tanto:

$$2.\int e^{x}.\text{sen } x.dx = e^{x}.\text{sen } x - e^{x}.\cos x \Rightarrow$$

$$\Rightarrow \int e^{x}.\text{sen } x.dx = \frac{e^{x}.\text{sen } x - e^{x}.\cos x}{2} + C$$

$\boxed{1.6.14}$

$$\int e^{3.x}.\cos 2.x.dx = \frac{e^{3.x}.\text{sen } 2.x}{2} - \frac{3}{2}.\int e^{3.x}.\text{sen } 2.x.dx =$$

$$\begin{array}{l} * \; u = e^{3.x} \Rightarrow du = 3.e^{3.x}.dx \\ * \; dv = \cos 2.x.dx \Rightarrow v = \int \cos 2.x.dx = \dfrac{\text{sen } 2.x}{2} \end{array}$$

$$= \frac{e^{3.x}.\text{sen } 2.x}{2} - \frac{3}{2}.(-\frac{e^{3.x}.\cos 2.x}{2} + \frac{3}{2}.\int e^{3.x}.\cos 2.x.dx)$$

$$\begin{array}{l} * \; u = e^{3.x} \Rightarrow du = 3.e^{3.x}.dx \\ * \; dv = \text{sen } 2.x.dx \Rightarrow v = \int \text{sen } 2.x.dx = -\dfrac{\cos 2.x}{2} \end{array}$$

O sea, aparece de nuevo la primitiva que debemos calcular:

$$\int e^{3.x}.\cos 2.x.dx = \frac{e^{3.x}.\text{sen } 2.x}{2} + \frac{3.e^{3.x}.\cos 2.x}{4} - \frac{9}{4}.\int e^{3.x}.\cos 2.x.dx$$

por tanto:

$$(1 + \frac{9}{4}).\int e^{3.x}.\cos 2.x.dx = \frac{e^{3.x}.\text{sen } 2.x}{2} + \frac{3.e^{3.x}.\cos 2.x}{4} \Rightarrow$$

$$\Rightarrow \int e^{3.x}.\cos 2.x.dx = \frac{4}{13}.(\frac{e^{3.x}.\text{sen } 2.x}{2} + \frac{3.e^{3.x}.\cos 2.x}{4}) + C$$

$\boxed{1.6.15}$

$$\int e^{5.x}.\text{sen } 3.x.dx = \frac{e^{5.x}.\text{sen } 3.x}{5} - \frac{3}{5}.\int e^{5.x}.\cos 3.x.dx =$$

$$\begin{array}{l} * \; u = \text{sen } 3.x \Rightarrow du = 3.\cos 3.x.dx \\ * \; dv = e^{5.x}.dx \Rightarrow v = \int e^{5.x}.dx = \dfrac{e^{5.x}}{5} \end{array}$$

$$= \frac{e^{5.x}.\text{sen } 3.x}{5} - \frac{3}{5}.(\frac{e^{5.x}.\cos 3.x}{5} + \frac{3}{5}.\int e^{5.x}.\text{sen } 3.x.dx)$$

$$\begin{array}{l} * \; u = \cos 3.x \Rightarrow du = -3.\text{sen } 3.x.dx \\ * \; dv = e^{5.x}.dx \Rightarrow v = \int e^{5.x}.dx = \dfrac{e^{5.x}}{5} \end{array}$$

O sea, aparece de nuevo la primitiva que debemos calcular:

$$\int e^{5.x}.\text{sen } 3.x.dx = \frac{e^{5.x}.\text{sen } 3.x}{5} - \frac{3.e^{5.x}.\cos 3.x}{25} - \frac{9}{25}.\int e^{5.x}.\text{sen } 3.x.dx$$

por tanto:

$$(1+\frac{9}{25}).\int e^{5.x}.\text{sen } 3.x.dx = \frac{e^{5.x}.\text{sen } 3.x}{5} - \frac{3.e^{5.x}.\cos 3.x}{25} \Rightarrow$$

$$\Rightarrow \int e^{5.x}.\text{sen } 3.x.dx = \frac{25}{34}.(\frac{e^{5.x}.\text{sen } 3.x}{5} - \frac{3.e^{5.x}.\cos 3.x}{25}) + C$$

En los siguientes cinco ejemplos calculamos primitivas de funciones cuya "estructura" es $f(x) = P(x).\text{Ln } Q(x)$, donde P(x) y Q(x) son polinomios. Para aplicar la formulita $\int u.dv = u.v - \int v.du$ haremos $u = \text{Ln } Q(x)$ y $dv = P(x).dx$, y no es raro que al final nos encontremos con el problema de calcular la primitiva de un cociente de polinomios.

$\boxed{1.6.16}$

$$\int \text{Ln } x.dx = x.\text{Ln } x - \int dx = x.\text{Ln } x - x + C$$

$* \; u = \text{Ln } x \Rightarrow du = dx/x$
$* \; dv = dx \Rightarrow v = \int dx = x$

$\boxed{1.6.17}$

$$\int x.\text{Ln } x.dx = \frac{x^2.\text{Ln } x}{2} - \frac{1}{2}.\int x.dx = \frac{x^2.\text{Ln } x}{2} - \frac{x^2}{4} + C$$

$* \; u = \text{Ln } x \Rightarrow du = dx/x$
$* \; dv = x.dx \Rightarrow v = \int x.dx = x^2/2$

$\boxed{1.6.18}$

$$\int x^2.\text{Ln } x.dx = \frac{x^3.\text{Ln } x}{3} - \frac{1}{3}.\int x^2.dx = \frac{x^3.\text{Ln } x}{3} - \frac{x^3}{9} + C$$

$* \; u = \text{Ln } x \Rightarrow du = dx/x$
$* \; dv = x^2.dx \Rightarrow v = \int x^2.dx = x^3/3$

$\boxed{1.6.19}$

$$\int \text{Ln } (1+x^2).dx = x.\text{Ln } (1+x^2) - 2.\int \frac{x^2}{1+x^2} dx = ?$$

$* \; u = \text{Ln } (1+x^2) \Rightarrow du = 2.x.dx/(1+x^2)$
$* \; dv = dx \Rightarrow v = \int dx = x$

Nuestro problema se transforma en el problema de calcular la primitiva de un cociente de polinomios; por tanto, hasta que no aprendamos a calcular este tipo de primitivas no podremos acabar el ejercicio.

$\boxed{1.6.20}$ $$\int x.\text{Ln}(1+x^2).dx = \frac{x^2.\text{Ln}(1+x^2)}{2} - 2.\int \frac{x^3}{1+x^2}dx = ?$$

$$* \ u = \text{Ln}(1+x^2) \Rightarrow du = 2.x.dx/(1+x^2)$$
$$* \ dv = x.dx \Rightarrow v = \int x.dx = x^2/2$$

Nuestro problema se transforma en el problema de calcular la primitiva de un cociente de polinomios; por tanto, hasta que no aprendamos a calcular este tipo de primitivas no podremos acabar el ejercicio.

En los siguientes seis ejemplos calculamos primitivas de funciones cuya "estructura" es $f(x) = P(x).T(x)$, donde P(x) es un polinomio y T(x) una función trigonométrica inversa. Para aplicar la formulita $\int u.dv = u.v - \int v.du$ haremos $u = T(x)$ y $dv = P(x).dx$, y no es raro que al final nos encontremos con el problema de calcular primitivas que de momento nos desbordan

$\boxed{1.6.21}$ $$\int \text{arc tg } x.dx = x.\text{arc tg } x - \int \frac{x}{1+x^2}.dx =$$

$$* \ u = \text{arc tg } x \Rightarrow du = dx/(1+x^2)$$
$$* \ dv = dx \Rightarrow v = \int dx = x$$

$$= x.\text{arc tg } x - \frac{1}{2}.\text{Ln}\left|1+x^2\right| + C$$

$\boxed{1.6.22}$ $$\int x.\text{arc tg } x.dx = \frac{x^2.\text{arc tg } x}{2} - \frac{1}{2}.\int \frac{x^2}{1+x^2}.dx = ?$$

$$* \ u = \text{arc tg } x \Rightarrow du = dx/(1+x^2)$$
$$* \ dv = dx \Rightarrow v = \int dx = x$$

Nuestro problema se transforma en el problema de calcular la primitiva de un cociente de polinomios; por tanto, hasta que no aprendamos a calcular este tipo de primitivas no podremos acabar el ejercicio.

$\boxed{1.6.23}$ $$\int \text{arc sen } x.dx = x.\text{arc sen } x - \int \frac{x}{\sqrt{1-x^2}}.dx =$$

$$* \ u = \text{arc sen } x \Rightarrow du = dx/\sqrt{1-x^2}$$
$$* \ dv = dx \Rightarrow v = \int dx = x$$

$$= x.\text{arc sen } x + \sqrt{1-x^2} + C$$

1.6.24 $\int x.\text{arc sec } x.dx = \frac{x^2.\text{arc sec } x}{2} - \frac{1}{2}.\int \frac{x}{\sqrt{x^2-1}}.dx =$

$$* \; u = \text{arc sec } x \Rightarrow du = \frac{dx}{x.\sqrt{x^2-1}}$$
$$* \; dv = x.dx \Rightarrow v = \int x.dx = x^2/2$$

$$= \frac{x^2.\text{arc sec } x}{2} - \frac{1}{2}.\sqrt{x^2-1} + C$$

1.6.25 $\int x.\text{arc sen } x.dx = \frac{x^2.\text{arc sen } x}{2} - \frac{1}{2}.\int \frac{x^2}{\sqrt{1-x^2}}.dx = ?$

$$* \; u = \text{arc sen } x \Rightarrow du = dx/\sqrt{1-x^2}$$
$$* \; dv = x.dx \Rightarrow v = \int x.dx = x^2/2$$

Nuestro problema se transforma en el problema de calcular la primitiva de una función de la forma $R(x;\sqrt{a^2-x^2})$; por tanto, hasta que no aprendamos a calcular este tipo de primitivas no podremos acabar el ejercicio.

1.6.26 $\int \text{arc sen } \frac{1}{x}.dx = x.\text{arc sen } \frac{1}{x} + \int \frac{dx}{\sqrt{x^2-1}} =$

$$* \; u = \text{arc sen } \frac{1}{x} \Rightarrow du = \frac{-1/x^2}{\sqrt{1-(1/x)^2}}.dx = -\frac{dx}{x.\sqrt{x^2-1}}$$
$$* \; dv = dx \Rightarrow v = \int dx = x$$

$$= x.\text{arc sen } \frac{1}{x} + \text{arg ch } x + C$$

$\boxed{1.6.27}$ $$\int \frac{dx}{\cos^4 x} = \int \frac{1}{\cos^2 x}.\frac{dx}{\cos^2 x} = \frac{tg\ x}{\cos^2 x} - \int \frac{2.sen\ x}{\cos^3 x}.tg\ x.dx =$$

$$\boxed{\begin{aligned} &* u = \frac{1}{\cos^2 x} \Rightarrow du = \frac{2.sen\ x}{\cos^3 x}.dx \\ &* dv = \frac{1}{\cos^2 x}.dx \Rightarrow v = \int \frac{1}{\cos^2 x}.dx = tg\ x \end{aligned}}$$

$$= \frac{tg\ x}{\cos^2 x} - 2.\int \frac{sen^2\ x}{\cos^4 x}.dx = \frac{tg\ x}{\cos^2 x} - 2.\int \frac{1-\cos^2 x}{\cos^4 x}.dx =$$

$$\boxed{sen^2\ x = 1 - \cos^2 x}$$

$$= \frac{tg\ x}{\cos^2 x} - 2.\underbrace{\int \frac{1}{\cos^4 x}.dx}_{\text{es la primitiva que buscamos}} + 2.\int \frac{\cos^2 x}{\cos^4 x}.dx$$

Resulta:

$$3.\int \frac{dx}{\cos^4 x} = \frac{tg\ x}{\cos^2 x} + 2.\int \frac{\cos^2 x}{\cos^4 x}.dx$$

o sea:

$$3.\int \frac{dx}{\cos^4 x} = \frac{tg\ x}{\cos^2 x} + 2.\int \frac{dx}{\cos^2 x} = \frac{tg\ x}{\cos^2 x} + 2.tg\,x$$

En definitiva:

$$\int \frac{dx}{\cos^4 x} = \frac{1}{3}.(\frac{tg\ x}{\cos^2 x} + 2.tg\,x) + C$$

$\boxed{1.6.28}$ $$\int x^3.(Ln\ x)^2.dx = \frac{x^4.(Ln\ x)^2}{4} - \frac{1}{2}.\int x^3.Ln\ x.dx =$$

$$\boxed{\begin{aligned} &* u = (Ln\ x)^2 \Rightarrow du = \frac{2.Ln\ x}{x}.dx \\ &* dv = x^3.dx \Rightarrow v = \int x^3.dx = x^4/4 \end{aligned}}$$

$$= \frac{x^4.(Ln\ x)^2}{4} - \frac{1}{2}.(\frac{x^4.Ln\ x}{4} - \frac{1}{4}.\int x^3.dx) =$$

$$\boxed{\begin{aligned} &* u = Ln\ x \Rightarrow du = \frac{dx}{x} \\ &* dv = x^3.dx \Rightarrow v = \int x^3.dx = x^4/4 \end{aligned}}$$

$$= \frac{x^4.(Ln\ x)^2}{4} - \frac{x^4.Ln\ x}{8} + \frac{1}{8}.\int x^3.dx =$$

$$= \frac{x^4.(Ln\ x)^2}{4} - \frac{x^4.Ln\ x}{8} + \frac{x^4}{32} + C$$

$\boxed{1.6.29}$ $\int x^2.\cos(\text{Ln } x).dx = \dfrac{x^3.\cos(\text{Ln } x)}{3} + \dfrac{1}{3}.\int x^2.\text{sen Ln } x.dx =$

$$* \; u = \cos(\text{Ln } x) \Rightarrow du = -\frac{\text{sen}(\text{Ln } x)}{x}.dx$$
$$* \; dv = x^2.dx \Rightarrow v = \int x^2.dx = x^3/3$$

$$= \frac{x^3.\cos(\text{Ln } x)}{3} + \frac{1}{3}.\left(\frac{x^3.\text{sen}(\text{Ln } x)}{3} - \frac{1}{3}.\underbrace{\int x^2.(\cos \text{Ln } x).dx}\right)$$

es la primitiva que buscamos

Por tanto:

$$\left(1 + \frac{1}{9}\right).\int x^2.\cos(\text{Ln } x).dx = \frac{x^3.\cos(\text{Ln } x)}{3} + \frac{x^3.\text{sen}(\text{Ln } x)}{9} \Rightarrow$$

$$\Rightarrow \int x^2.\cos(\text{Ln } x).dx = \frac{9}{10}.\left(\frac{x^3.\cos(\text{Ln } x)}{3} + \frac{x^3.\text{sen}(\text{Ln } x)}{9}\right) + C$$

$\boxed{1.6.30}$ $\int 2.x.\text{sen}^2 x.dx = \int 2.x.\dfrac{1-\cos 2.x}{2}.dx = \int x.(1-\cos 2.x).dx =$

Todo el mundo sabe que $\text{sen}^2 x = \dfrac{1-\cos 2.x}{2}$

$$= \int x.dx - \int x.\cos 2.x.dx = \frac{x^2}{2} - \int x.\cos 2.x.dx =$$

$$= \frac{x^2}{2} - \left(\frac{x.\text{sen } 2.x}{2} - \frac{1}{2}.\int \text{sen } 2.x.dx\right) =$$

$$* \; u = x \Rightarrow du = dx$$
$$* \; dv = \cos 2.x.dx \Rightarrow v = \int \cos 2.x.dx = \frac{\text{sen } 2.x}{2}$$

$$= \frac{x^2}{2} - \frac{x.\text{sen } 2.x}{2} - \frac{\cos 2.x}{4} + C$$

$\boxed{1.6.30}$ $\int 2.x.\cos^2 x.dx = \int 2.x.\dfrac{1+\cos 2.x}{2}.dx = \int x.(1+\cos 2.x).dx =$

Todo el mundo sabe que $\cos^2 x = (1+\cos 2.x)/2$

$$= \int x.dx + \int x.\cos 2.x.dx = \frac{x^2}{2} + \int x.\cos 2.x.dx =$$

$$* \; u = x \Rightarrow du = dx$$
$$* \; dv = \cos 2.x.dx \Rightarrow v = \int \cos 2.x.dx = \frac{\text{sen } 2.x}{2}$$

$$= \frac{x^2}{2} + \left[\frac{x.\text{sen } 2.x}{2} - \frac{1}{2}.\int \text{sen } 2.x.dx \right] =$$

$$= \frac{x^2}{2} + \frac{x.\text{sen } 2.x}{2} + \frac{\cos 2.x}{4} + C$$

$\boxed{1.6.31}$ $$\int x^3.e^{x^2}.dx = \int x^2.x.e^{x^2}.dx = \frac{x^2.e^{x^2}}{2} - \int x.e^{x^2}.dx =$$

$$\boxed{\begin{array}{c} * \; u = x^2 \Rightarrow du = 2.x.dx \\ * \; dv = x.e^{x^2}.dx \Rightarrow v = \int x.e^{x^2}.dx = \frac{e^{x^2}}{2} \end{array}}$$

$$= \frac{x^2.e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$$

$\boxed{1.6.32}$ $$\int (\cos x).\text{Ln}\,(1+\cos x).dx =$$

$$= (\text{sen } x).\text{Ln}\,(1+\cos x) + \int \frac{\text{sen}^2 x}{1+\cos x}.dx =$$

$$\boxed{\begin{array}{c} * \; u = \text{Ln}\,(1+\cos x) \Rightarrow du = -\frac{\text{sen } x}{1+\cos x}.dx \\ * \; dv = \cos x.dx \Rightarrow v = \int \cos x.dx = \text{sen } x \end{array}}$$

$$= (\text{sen } x).\text{Ln}\,(1+\cos x) + \int (1-\cos x).dx =$$

$$\boxed{\frac{\text{sen}^2 x}{1+\cos x} = \frac{1-\cos^2 x}{1+\cos x} = \frac{(1+\cos x).(1-\cos x)}{1+\cos x} = 1-\cos x}$$

$$= (\text{sen } x).\text{Ln}\,(1+\cos x) + x - \text{sen } x + C$$

$\boxed{1.6.33}$ $$\int x^n.\text{Ln } x.dx = \frac{x^{n+1}}{n+1}.\text{Ln } x - \frac{1}{n+1}.\int x^n.dx =$$

$$\boxed{\begin{array}{c} * \; u = \text{Ln } x \Rightarrow du = \frac{dx}{x} \\ * \; dv = x^n.dx \Rightarrow v = \int x^n.dx = \frac{x^{n+1}}{n+1} \end{array}}$$

$$= \frac{x^{n+1}}{n+1}.\text{Ln } x - \frac{x^{n+1}}{(n+1)^2} + C$$

$\boxed{1.6.34}$ $$\int \frac{\text{Ln } x}{x^2}.dx = -\frac{\text{Ln } x}{x} + \int \frac{dx}{x^2} = -\frac{\text{Ln } x}{x} - \frac{1}{x} + C$$

$$\boxed{\begin{array}{c} * \; u = \text{Ln } x \Rightarrow du = \frac{dx}{x} \\ * \; dv = dx/x^2 \Rightarrow v = \int dx/x^2.dx = -1/x \end{array}}$$

1.7 CAMBIO DE VARIABLE

Si para calcular $\int f(x).dx$ efectuamos el cambio de variable definido por:

$$x = \phi(t) \Rightarrow dx = \phi'(t).dt$$

Al sustituir "x" por $\phi(t)$ y "dx" por $\phi'(t).dt$, se obtiene:

$$\int f(x).dx = \int f(\phi(t)).\phi'(t).dt$$

Y ahora ***rezamos*** para que el problema de calcular $\int f(\phi(t)).\phi'(t).dt$ sea más sencillo que el problema de calcular $\int f(x).dx$.

No hay receta que en todos los casos diga cuál es el cambio de variable que resuelve la papeleta; pero don't worry, no van a pedirte que descubras cambios de variable hasta ahora no descubiertos. Sólo te pedirán que conozcas los cambios de variable más famosos; de ellos hablaremos en su momento

1.7.1

$$\int \cos 3.x.dx = \int \cos z.\frac{dz}{3} = \frac{1}{3}.\text{sen } z =$$

Aunque la primitiva es inmediata, también la podemos calcular haciendo el cambio $3.x = z \Rightarrow x = z/3 \Rightarrow dx = dz/3$

$$= \frac{1}{3}.\text{sen } 3.x + C$$

deshacemos el cambio de variable; o sea, sustituimos "z" por "3.x"

1.7.2

$$\int \frac{dx}{(x+4)^6} = \int \frac{dx}{z^6} = \frac{z^{-6+1}}{-6+1} = -\frac{1}{5.z^5} =$$

Aunque la primitiva es inmediata, también la podemos calcular haciendo el cambio $x + 4 = z \Rightarrow dx = dz$

$$= -\frac{1}{5.(x+4)^5} + C$$

deshacemos el cambio de variable; o sea, sustituimos "z" por "x + 4"

1.7.3

$$\int x.\sqrt{x-1}.dx = \int (1+z^2).z.(2.z.dz) = 2.\int (z^2+z^4).dz =$$

$\sqrt{x-1} = z \Rightarrow x - 1 = z^2 \Rightarrow x = 1 + z^2 \Rightarrow dx = 2.z.dz$

$$= 2.(\frac{1}{3}.z^3 + \frac{1}{5}.z^5) = \frac{2}{3}.(\sqrt{x-1})^3 + \frac{2}{5}.(\sqrt{x-1})^5 + C$$

deshacemos el cambio de variable; o sea, sustituimos "z" por $\sqrt{x-1}$

$\boxed{1.7.4}$

$$\int \frac{2.x+3}{(x-1)^2+4^2}.dx = \int \frac{2.(1+4.t)+3}{4^2.t^2+4^2}.4.dt =$$

$$\boxed{x-1=4.t \Rightarrow x=1+4.t \Rightarrow dx=4.dt}$$

$$= \frac{5}{4}.\int \frac{dt}{t^2+1} + \int \frac{2.t.dt}{t^2+1} = \frac{5}{4}.\text{arc tg } t + \text{Ln}\left|t^2+1\right| =$$

$$= \frac{5}{4}.\text{arc tg } \frac{x-1}{4} + \text{Ln}\left|(\frac{x-1}{4})^2+1\right| + C$$

$$\boxed{\text{deshacemos el cambio de variable: } x-1=4.t \Rightarrow t=\frac{x-1}{4}}$$

$\boxed{1.7.5}$

$$\int \frac{3.x-2}{(x+1)^2+5^2}.dx = \int \frac{3.(-1+5.t)-2}{5^2.t^2+5^2}.5.dt =$$

$$\boxed{x+1=5.t \Rightarrow x=-1+5.t \Rightarrow dx=5.dt}$$

$$= -\int \frac{dt}{t^2+1} + 3.\int \frac{t.dt}{t^2+1} = -\text{arc tg } t + \frac{3}{2}.\text{Ln}\left|t^2+1\right| =$$

$$= -\text{arc tg } \frac{x+1}{5} + \frac{3}{2}.\text{Ln}\left|(\frac{x+1}{5})^2+1\right| + C$$

$$\boxed{\text{deshacemos el cambio de variable: } x+1=5.t \Rightarrow t=\frac{x+1}{5}}$$

$\boxed{1.7.6}$

$$\int \cos\sqrt{x}.dx = \int (\cos y).(2.y.dy) = 2.\int y.\cos y.dy =$$

$$\boxed{\sqrt{x}=y \Rightarrow x=y^2 \Rightarrow dx=2.y.dy}$$

$$= 2.(y.\text{sen } y - \int \text{sen } y.dy) = 2.(y.\text{sen } y + \cos y) =$$

$$\boxed{\begin{array}{c} * \; u=y \Rightarrow du=dy \\ * \; dv=\cos y.dy \Rightarrow v=\int \cos y.dy = \text{sen } y \end{array}}$$

$$= 2.(\sqrt{x}.\text{sen }\sqrt{x} + \cos\sqrt{x}) + C$$

$$\boxed{\text{deshacemos el cambio de variable: } y=\sqrt{x}}$$

$\boxed{1.7.7}$

$$\int \text{arc sen } x.dx = \int y.\cos y.dy =$$

$$\boxed{\text{arc sen } x = y \Rightarrow x=\text{sen } y \Rightarrow dx=\cos y.dy}$$

$$= y.\text{sen } y + \cos y =$$

$$\boxed{\text{calculada en el ejercicico anterior}}$$

$$= (\text{arc sen } x).(\text{sen arc sen } x) + \cos \text{arcsen} x + C =$$

$$\boxed{\text{deshacemos el cambio de variable: } y=\text{arc sen } x}$$

$$= x.\text{arc sen } x + \sqrt{1-x^2} + C$$

$$* \text{ es sen arc sen } x = x$$

$$* \text{ es cos arc sen } x = \cos y = \sqrt{1-\text{sen}^2 y} = \sqrt{1-x^2}$$

$$\boxed{\text{arc sen } x = y} \qquad \boxed{\text{sen } y = x}$$

$\boxed{1.7.8}$

$$\int \frac{x}{\sqrt{1+x}}.dx = \int \frac{z^2-1}{z}.(2.z.dz) =$$

$$\boxed{\sqrt{1+x} = z \Rightarrow 1+x = z^2 \Rightarrow x = z^2-1 \Rightarrow dx = 2.z.dz}$$

$$= 2.\int (z^2-1).dz = \frac{2}{3}.z^3 - \frac{1}{2}.z = \frac{2}{3}.(\sqrt{1+x}) - \frac{1}{2}.\sqrt{1+x} + C$$

$$\boxed{\text{deshacemos el cambio de variable: } z = \sqrt{1+x}}$$

$\boxed{1.7.9}$

$$\int \sqrt{9-x^2}.dx = \int \sqrt{9-9.\text{sen}^2 z}.3.\cos z.dz =$$

$$\boxed{x = 3.\text{sen } z \Rightarrow dx = 3.\cos z.dz}$$

$$= 9.\int \sqrt{1-\text{sen}^2 z}.\cos z.dz = 9.\int \cos^2 z.dz =$$

$$\boxed{\sqrt{1-\text{sen}^2 z} = \cos z}$$

$$= 9.\int \frac{1+\cos 2.z}{2}.dz = \frac{9}{2}.z + \frac{9}{4}.\text{sen } 2.z =$$

$$\boxed{\text{Todo el mundo sabe que } \cos^2 z = \frac{1+\cos 2.z}{2}}$$

$$= \frac{9}{2}.\text{arc sen } \frac{x}{3} + \frac{9}{4}.\frac{2.x}{3}.\sqrt{1-(x/3)^2} + C$$

Deshacemos el cambio de variable:

$$* \; x = 3.\text{sen } z \Rightarrow \text{sen } z = \frac{x}{3} \Rightarrow z = \text{arc sen } \frac{x}{3}$$

$$* \; \text{sen } 2.z = 2.\text{sen } z.\cos z = \frac{2.x}{3}.\sqrt{1-(x/3)^2}$$

$$\boxed{\text{sen } z = \frac{x}{3} \; ; \; \cos z = \sqrt{1-\text{sen}^2 z} = \sqrt{1-(x/3)^2}}$$

$\boxed{1.7.10}$

$$\int \frac{1}{1+e^x}.dx = \int \frac{1}{1+e^{-\text{Ln } z}}.(-\frac{dz}{z}) =$$

$$\boxed{x = -\text{Ln } z \Rightarrow dx = -\frac{dz}{z}}$$

$$= \int \frac{1}{1+(1/z)}.(-\frac{dz}{z}) = -\int \frac{dz}{z+1} = -Ln|z+1| = -Ln|e^{-x}+1| + C$$

$\boxed{e^{-Ln\ z} = 1/z}$

$\boxed{\text{deshacemos el cambio de variable: } x = -Ln\ z \Rightarrow Ln\ z = -x \Rightarrow z = e^{-x}}$

1.7.11 $$\int \cos^5 x.dx = \int \cos^4 x.\cos x.dx = \int (1-\operatorname{sen}^2 x)^2.\cos x.dx =$$

$$= \int (1-z^2)^2.dz = \int (1-2.z^2+z^4).dz = z - \frac{2}{3}.z^3 + \frac{1}{5}.z^5 =$$

$\boxed{\operatorname{sen} x = z \Rightarrow \cos x.dx = dz}$

$$= \operatorname{sen} x - \frac{2}{3}.\operatorname{sen}^3 x + \frac{1}{5}.\operatorname{sen}^5 x + C$$

$\boxed{\text{deshacemos el cambio de variable: } z = \operatorname{sen} x}$

1.7.12 $$\int \operatorname{sen}^5 x.dx = \int \operatorname{sen}^4 x.\operatorname{sen} x.dx = \int (1-\cos^2 x)^2.\operatorname{sen} x.dx =$$

$$= -\int (1-z^2)^2.dz = -\int (1-2.z^2+z^4).dz = -z + \frac{2}{3}.z^3 - \frac{1}{5}.z^5 =$$

$\boxed{\cos x = z \Rightarrow -\operatorname{sen} x.dx = dz}$

$$= -\cos x + \frac{2}{3}.\cos^3 x - \frac{1}{5}.\cos^5 x + C$$

$\boxed{\text{deshacemos el cambio de variable: } z = \cos x}$

1.7.13 $$\int \frac{(Ln\ x)^2 + (Ln\ x)^3 + (Ln\ x)^4}{x}.dx = \int (z^2+z^3+z^4).dz =$$

$\boxed{Ln\ x = z \Rightarrow dx/x = dz}$

$$= \frac{1}{3}.z^3 + \frac{1}{4}.z^4 + \frac{1}{5}.z^5 =$$

$$= \frac{1}{3}.(Ln\ x)^3 + \frac{1}{4}.(Ln\ x)^4 + \frac{1}{5}.(Ln\ x)^5 + C$$

$\boxed{\text{deshacemos el cambio de variable: } z = Ln\ x}$

1.7.14 $$\int e^{\sqrt{x}}.dx = \int e^z.2.z.dz =$$

$\boxed{\sqrt{x} = z \Rightarrow x = z^2 \Rightarrow dx = 2.z.dz}$

$$\boxed{\begin{array}{c} * \ u = z \Rightarrow du = dz \\ * \ dv = e^z.dz \Rightarrow v = \int e^z.dz = e^z \end{array}}$$

$$= 2.(z.e^z - \int e^z.dz) = 2.(z.e^z - e^z) = 2.(\sqrt{x}.e^{\sqrt{x}} - e^{\sqrt{x}}) + C$$

$\boxed{\text{deshacemos el cambio de variable: } z = \sqrt{x}}$

$\boxed{1.7.15}$

$$\int \frac{dx}{(x-3).\sqrt{x^2-6.x+5}} = \int \frac{dx}{(x-3).\sqrt{(x-3)^2-4}} =$$

astutamente, observamos que $x^2 - 6.x + 5 = (x-3)^2 - 4$

$$= \int \frac{1}{\frac{1}{z}.\sqrt{\frac{1}{z^2}-4}}.(-\frac{1}{z^2}.dz) =$$

padecemos una hemorragia de genialidad y hacemos el cambio $x - 3 = 1/z \Rightarrow dx = -dz/z^2$

$$= -\int \frac{dz}{\sqrt{1-4.z^2}} = -\frac{1}{2}.\text{arc sen } 2.z =$$

$$= -\frac{1}{2}.\text{arc sen } \frac{2}{x-3} + C$$

deshacemos el cambio de variable: $x - 3 = 1/z \Rightarrow z = 1/(x-3)$

$\boxed{1.7.16}$

$$\int \cos \sqrt[3]{x}.dx = \int (\cos z).(3.z^2.dz) = 3.\int z^2.\cos z.dz =$$

$\sqrt[3]{x} = z \Rightarrow x = z^3 \Rightarrow dx = 3.z^2.dz$

hiperfamosa: por partes dos veces

$$= \ .... \ = g(z) = g(\sqrt[3]{x}) + C$$

deshacemos el cambio de variable: $z = \sqrt[3]{x}$